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On the Planar Piecewise Quadratic 1-Center Problem

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Abstract In this paper we introduce a minimax model unifying several classes of single facility planar center location problems. We assume that the transportation costs of the demand points to the serving facility are convex functions $\{Q_i\}$, $i = 1, \ldots, n$, of the planar distance used. Moreover, these functions, when properly transformed, give rise to piecewise quadratic functions of the coordinates of the facility location. In the continuous case, using results on LP-type models by Clarkson (J. ACM 42:488–499, 1995), Matoušek et al. (Algorithmica 16:498–516, 1996), and the derandomization technique in Chazelle and Matoušek (J. Algorithms 21:579–597, 1996), we claim that the model is solvable deterministically in linear time. We also show that in the separable case, one can get a direct $O(n \log n)$ deterministic algorithm, based on Dyer (Proceedings of the 8th ACM Symposium on Computational Geometry, 1992), to find an optimal solution. In the discrete case, where the location of the center (server) is restricted to some prespecified finite set, we introduce deterministic subquadratic algorithms based on the general parametric approach of Megiddo (J. ACM 30:852–865, 1983), and on properties of upper envelopes of collections of quadratic arcs. We

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apply our methods to solve and improve the complexity of a number of other location problems in the literature, and solve some new models in linear or subquadratic time complexity.

Keywords Center location · Quadratic programming · LP-type models · Parametric approach

1 Introduction

Since the early seventies, single facility geometric minimax location models have been discussed and analyzed quite frequently from a mathematical programming point of view. Megiddo [30], in his 1983 seminal paper, was the first to prove that the classical planar minimum covering sphere problem (studied algorithmically by Elzinga and Hearn [19, 20]), can be solved in optimal linear time as a 3-variable convex quadratic program. Megiddo's work has since been extended to solve in linear time other related, more general, geometric problems, viewed as convex mathematical programs. (Representative examples are the weighted Euclidean center problem [17], the minimum volume covering ellipsoid [18], and the minimum ball spanned by balls [33], in any fixed dimension). The above models are special cases of a minimax single facility (1-center) geometric location model where the transportation costs of the customers are convex quadratic functions of the location of the server (facility).

We also find in the literature several papers dealing with related minimax geometric problems where the transportation cost functions are not pure quadratic, e.g., the round trip 1-center problem [9, 15], the quadratic bicriteria model in [36] and the stochastic rectilinear 1-center problem [22]. These models are currently solved in non-optimal superlinear time.

In the above (continuous) geometric models, the server can be located at any point in the plane (or in some convex subset). The discrete versions of these models, where the server is constrained to belong to some discrete prespecified subset, have also been extensively studied. Note that, unlike the continuous models, optimal algorithms for some of the discrete models have superlinear complexity, e.g., the discrete minimum covering sphere problem [28].

Our goal in this paper is to present a convex, piecewise quadratic model which generalizes and unifies a variety of planar 1-center problems in the literature. We will show that the continuous version of the model can be solved in optimal linear time by using the modern LP-type algorithms developed by Clarkson [12] and Chazelle and Matoušek [10]. For the discrete version of our unifying model we present a subquadratic algorithm based on the general parametric approach of Megiddo [31], and on properties of upper envelopes of collections of quadratic arcs.

To further motivate and explain the features of our unifying general model, we first provide a more detailed survey of some of the most common related models in the literature.

Given a set of n points in the plane, $V = \{v_i = (A_i, B_i) : i = 1, \dots, n\}$, viewed as demand points, the planar 1-center problem is to find the location $x = (x_1, x_2)$ of a single server (center), minimizing the maximum of the transportation costs from the demand points to the server. Formally, if d denotes the planar distance function used, each demand point v_i, $i = 1, \dots, n$, is associated with a monotone real function $Q_i(t)$, and the transportation cost function from v_i to a server located at $x = (x_1, x_2)$ is given by $Q_i(d(v_i, x))$. The goal is to minimize $\max_{i=1,\dots,n} Q_i(d(v_i, x))$. The reader is referred to the two papers by Frenk, Gromicho and Zhang [23, 24] for planar 1-center problems, involving general nonlinear transportation cost functions. (Even though our study focuses on piecewise quadratic models, in Sect. 6.5 we show some non quadratic problems that can be solved by the techniques presented in the paper.)

The case where all the functions $\{Q_i\}$ are linear unifies a variety of classical Euclidean geometric problems. For example, the Euclidean 1-center problem, where $Q_i(d(v_i, x)) = d(v_i, x)$ for $i = 1, \dots, n$, and d is the Euclidean distance, amounts to finding the circle of minimum radius enclosing all the points in V. This problem was optimally solved in $O(n)$ time in the seminal paper of Megiddo [30]. Two other more general classical geometric models are defined as follows: Given is a set of n discs $\{D_i : i = 1, \dots, n\}$ in the plane. For $i = 1, \dots, n$, D_i has a radius r_i and it is centered at v_i. In the first model [33], the goal is to find a disc of minimum radius enclosing the n given discs. It corresponds to the case where $Q_i(d(v_i, x)) = r_i + d(v_i, x)$, for $i = 1, \dots, n$. In the second model, the goal is to find a disc of minimum radius intersecting the n given discs. This problem corresponds to the case where $Q_i(d(v_i, x)) = -r_i + d(v_i, x)$, for $i = 1, \dots, n$. (We are unaware of any paper discussing the second model explicitly. We note, however, that in this facility location problem, each disc D_i, $i = 1, \dots, n$, models an 'extensive' customer, and the transportation cost is the distance of the server from the extensive customer. See [37] for more examples of location problems dealing with extensive customers, e.g., paths, subtrees or neighborhoods in a network metric space.) Dyer [17] generalized the result in [30], and presented an $O(n)$ algorithm for solving the weighted Euclidean 1-center model defined by setting $Q_i(d(v_i, x)) = w_i d(v_i, x)$, $i = 1, \dots, n$. (The weights $\{w_i\}_i$ are assumed to be positive.)

Finally, Dyer [18] unified all the above Euclidean models and presented a linear time algorithm for the case defined by $Q_i(d(v_i, x)) = s_i + w_i d(v_i, x)$, for $i = 1, \dots, n$, where $\{s_i\}_i$ are arbitrary reals and $\{w_i\}_i$ are nonnegative weights. (Dyer considered only the case where $\{s_i\}_i$ are nonnegative. Nevertheless, his model can be easily adapted to the general case. See the general formulation in Sect. 6.1.) In particular, it follows that even the following Euclidean geometric problem is solvable in $O(n)$ time: Given the above set of n discs, $\{D_i : i = 1, \dots, n\}$, and an integer $1 \le n' \le n$, find a disc of minimum radius, enclosing (containing) the n' discs $\{D_i : i = 1, \dots, n'\}$ and intersecting the $n - n'$ discs $\{D_i : i = n' + 1, \dots, n\}$.

We note in passing that the rectilinear (ℓ_1) versions of the above Euclidean classical models are reducible to 3-variable linear programs, and therefore can be solved in $O(n)$ time [30], by using the following LP formulation.

$$\begin{aligned} \min \quad & z, \\ \text{s.t.} \quad & z \ge s_i + w_i(A_i - x_1) + w_i(B_i - x_2), \quad \forall i = 1, \dots, n, \\ & z \ge s_i - w_i(A_i - x_1) + w_i(B_i - x_2), \quad \forall i = 1, \dots, n, \\ & z \ge s_i + w_i(A_i - x_1) - w_i(B_i - x_2), \quad \forall i = 1, \dots, n, \end{aligned}$$

$$z \geq s_i - w_i(A_i - x_1) - w_i(B_i - x_2), \quad \forall i = 1, \ldots, n.$$

In comparison, to solve the above weighted Euclidean problems in linear time, Dyer [18] presented a simple and clever way to formulate these problems as fixed dimensional "almost" linear programs. (See the Appendix.) Specifically, in these convex programs the objective function and all but a fixed number of constraints are linear, and the nonlinear constraints are convex and "essentially equivalent" to polynomials. He also showed how to solve this class of convex programs in $O(n)$ time. In the context of the planar 1-center problem defined above, the model and algorithm in [18] do not seem to be applicable to location problems where the cost functions $\{Q_i\}_i$, when expressed in terms of the variables (x_1, x_2), are neither separable nor representable as pure convex quadratics. Our study was motivated by one such problem where the cost functions are separable but consist of several convex quadratic pieces. This is the following probabilistic planar center problem discussed and analyzed recently by Foul [22].

Let $\{Y_i = (U_i, V_i) : i = 1, \ldots, n\}$, be a set of n independent bivariate random variables. For $i = 1, \ldots, n$, $U_i(V_i)$, is a uniform random variable in the interval $[A_i - r_i/2, A_i + r_i/2]$, $([B_i - s_i/2, B_i + s_i/2])$, $r_i > 0$, $(s_i > 0)$, with probability density function $f_{U_i}(.)(f_{V_i}(.))$. The set $\{Y_i\}_i$ represents a set of n random demand points in the plane, where the expected value of the random vector Y_i is $E[Y_i] = (A_i, B_i)$, for $i = 1, \ldots, n$. For each pair of points $x = (x_1, x_2)$, $y = (y_1, y_2)$ in the plane let $d_1(y, x)$ denote the rectilinear planar distance between them. The *rectilinear planar weighted* 1*-center problem with uniformly distributed demand points* is to find a point $x^* = (x_1^*, x_2^*)$ minimizing the maximum of the weighted expected distances from x to the n demand points. Specifically, minimize the nonlinear function

$$U(x) = U(x_1, x_2) = \max_{i=1,\ldots,n} w_i E[d_1(Y_i, x)].$$

w_i, $i = 1, \ldots, n$, is a positive real weight. Restricting the domain of $U(x_1, x_2)$ to the rectangle $R_0 = \{(x_1, x_2) : A' \leq x_1 \leq A'', B' \leq x_2 \leq B''\}$, where $A' = \min_{i=1,\ldots,n}(A_i - r_i/2)$, $B' = \min_{i=1,\ldots,n}(B_i - s_i/2)$, $A'' = \max_{i=1,\ldots,n}(A_i + r_i/2)$, and $B'' = \max_{i=1,\ldots,n}(B_i + s_i/2)$, it is shown in [22] that

$$U(x_1, x_2) = \max_{i=1,\ldots,n} w_i(f_i(x_1) + g_i(x_2)),$$

where

$$f_i(x_1) = \begin{cases} (x_1 - A_i)^2/r_i + r_i/4, & \text{if } A_i - r_i/2 \leq x_1 \leq A_i + r_i/2, \\ -x_1 + A_i, & \text{if } A' \leq x_1 \leq A_i - r_i/2, \\ x_1 - A_i, & \text{if } A_i + r_i/2 \leq x_1 \leq A'', \end{cases}$$

and

$$g_i(x_2) = \begin{cases} (x_2 - B_i)^2/s_i + s_i/4, & \text{if } B_i - s_i/2 \leq x_2 \leq B_i + s_i/2, \\ -x_2 + B_i, & \text{if } B' \leq x_2 \leq B_i - s_i/2, \\ x_2 - B_i, & \text{if } B_i + s_i/2 \leq x_2 \leq B''. \end{cases}$$

Looking at the expression of $U(x_1, x_2)$ we note that this rectilinear probabilistic model is a special case of a more general model, where the transportation cost of the i-th demand point is given by a bivariate real function

$$Q_i(|x_1 - A_i|, |x_2 - B_i|), \tag{1}$$

where Q_i is a separable, convex and piecewise quadratic function. Foul [22] suggests the use of classical infinite iterative nonlinear 2-dimensional search procedures to approximate the optimal solution within any desirable accuracy level. We will show that this model can actually be solved exactly by a linear time deterministic algorithm.

More generally, in this paper we concentrate on a model unifying the above and several other classes of planar center location problems where the functions $\{Q_i\}_i$ give rise to convex and piecewise quadratic functions of the variables (x_1, x_2).

As a planar convex program this problem is an LP-type (or GLP) problem, and therefore can also be solved by randomized algorithms which require only a linear number of operations. (See [1, 2, 12, 29].) Moreover, since all the functions are convex piecewise quadratic, (with a constant number of pieces per each function), the derandomization technique of the algorithm of Clarkson [12] presented in [10], is applicable to this model, yielding a deterministic $O(n)$ time algorithm.

We also consider the discrete version of our general planar continuous model, where the location of the center (server) is constrained to be in some prespecified finite set of cardinality m. The above linear time methods of Dyer [18], Clarkson [12], Matoušek, Sharir and Welzl [29], and Chazelle and Matoušek [10], are not applicable to the discrete models. In fact, as noted above, even the classical discrete Euclidean planar 1-center problem, where $m = n$, has an optimal nonlinear $\theta(n \log n)$ algorithm [28]. Halman [26, 27] developed a discrete LP-type model, and presented a randomized linear time algorithm for its solution. With his framework the classical discrete weighted rectilinear center problem with $m = n$, can be solved by a randomized algorithm in (expected) linear time. (The best known complexity result for a deterministic algorithm for this discrete weighted problem is still $O(n \log n)$.) However, his framework is not applicable to the discrete version of our general planar model. In fact, as shown in [26], his discrete model is not applicable even to the Euclidean planar center problem. A crucial finiteness assumption on the dimension of certain discrete Helly systems is not satisfied in the Euclidean case.

To solve the discrete planar version, we will introduce deterministic subquadratic algorithms based on the general parametric approach of Megiddo [31], and on properties of upper envelopes of collections of quadratic arcs. (We note that this latter parametric approach is also applicable to the continuous models. Its complexity is superlinear, higher than the linear complexity of the above LP-type approach. Nevertheless, in situations where we need to solve simultaneously, both, the continuous and the discrete cases, e.g., in feasibility studies, the parametric approach might be advantageous.)

The methods we suggest solve and improve the complexity of a number of location problems in the literature, and solve some new models in linear or subquadratic time complexity. Tables 1 and 2 present a summary of the general results included in the paper. (Unless otherwise stated, the complexity bounds in the tables refer to deterministic algorithms.) The implementation of these results to specific models from the literature appear in Sect. 6.

Table 1 Complexity results for the continuous problem

	Weighted continuous 1-center			
	Extended parametric	Piecewise quadratic	Euclidean	Rectilinear
$\mathbb{R}^2$	Theorem 4.2 $O(\lambda_5(n)\log^3 n)$ $O(n)$ (randomized)	Theorem 3.1 $O(n)$	[17] $O(n)$	[32] $O(n)$
$\mathbb{R}^d$	Proposition 5.2 $O(n)$ (randomized)	Proposition 5.1 $O(n)$	[17] $O(n)$	[32] $O(n)$

Table 2 Complexity results for the discrete problem

	Weighted discrete 1-center		
	Extended parametric	Euclidean	Rectilinear
$\mathbb{R}^2$	Theorem 4.2 $O((m+\lambda_5(n))\log^3 n)$	Theorem 4.1 $O(n\log^3 n)$	Proposition 5.3 $O(n\log n)$
$\mathbb{R}^d$			Proposition 5.3 $O(n\log n)$ [27] $O(n)$ (randomized)

The paper is organized as follows. In Sect. 2 we present a general minimax planar optimization problem using continuous, convex, piecewise quadratic functions. In Sect. 3 we discuss the solvability of the continuous model. Specifically, we note that the linear time deterministic algorithms of Chazelle and Matoušek [10] are applicable. In the separable case there is an alternative algorithm to solve the continuous model. In the pure quadratic case the deterministic $O(n)$ geometric algorithm of Dyer [18], which seems to be simpler than the $O(n)$ method of Chazelle and Matoušek [10], is applicable. (In the piecewise quadratic case one can also find the exact optimal solution in $O(n\log n)$ time, by implementing the algorithm in [18], after solving $O(\log n)$ line restricted subproblems.)

In Sect. 4 we focus on the discrete models. We combine methods of computing envelopes of algebraic curves, with the parametric approach of Megiddo [31] to produce subquadratic algorithms.

Section 5 is devoted to the extensions of our results to $\mathbb{R}^d$. In Sect. 6 we present several models from the literature that can be solved using the methodologies developed in previous sections. The paper ends with some final comments and questions. We also include an Appendix that contains, for the sake of readability, some technical results that are used in the paper.

2 The Unifying Model

We consider the following minimax planar optimization problem, generalizing and unifying several classes of convex single facility planar location problems. Let

$R_0 \subseteq \mathbb{R}^2$ be a closed and bounded rectangle. For $i = 1, \dots, n$, let $H_i(x_1, x_2)$ be a continuous, convex, piecewise quadratic function, where its pieces are induced by a planar grid (arrangement) defined by p_i lines, which we call *partitioning lines*. p_i is assumed to be constant. If $p_i = 0$ we will say that $H_i(x_1, x_2)$ is a *pure quadratic*. (A quadratic function $q(x)$ on $\mathbb{R}^d$ is defined by $q(x) = x^T Dx + c^T x + k$, where D is a symmetric real matrix, c is a vector and k is a real. $q(x)$ is convex if and only if D is also positive semi-definite.)

$$\begin{aligned} \min \quad & U(x_1, x_2) = \max\{H_i(x_1, x_2) : i = 1, \dots, n\}, \\ \text{s.t.} \quad & (x_1, x_2) \in R_0. \end{aligned} \tag{2}$$

Note that $U(x_1, x_2)$ is the upper envelope (pointwise maximum function) of the collection $\{H_i\}_i$, and therefore it is convex and piecewise quadratic. We refer to the above problem as *the (continuous) planar piecewise quadratic* 1*-center problem*. An equivalent parametric formulation is given by,

$$\begin{aligned} \min \quad & z, \\ \text{s.t.} \quad & H_i(x_1, x_2) \le z, \quad \forall i = 1, \dots, n, \\ & (x_1, x_2) \in R_0. \end{aligned}$$

Since the dependence on the parameter z is linear, we will sometimes refer to the problem as the *linear parametric problem*. We also focus on the following discrete version of the above 1-center problem, where there is an additional constraint, restricting the location of the center to some prespecified planar finite set. Formally, given is a set of m points $V = \{v_j = (A_j, B_j) : j = 1, \dots, m\}$ in the plane. The *discrete planar piecewise quadratic* 1*-center problem* is formulated as

$$\min_{j=1,\dots,m} U(A_j, B_j).$$

Throughout the paper we will also discuss the particular geometric instances of our model corresponding to the Euclidean and rectilinear 1-center problems. As noted above, these cases are defined by setting $Q_i(d(x, v_i)) = w_i d(x, v_i) + s_i$ for $i = 1, \dots, n$. We refer to the terms $\{s_i\}_i$ as *addends*.

Consider the Euclidean case. If $s_i = 0$ for all $i = 1, \dots, n$, then by setting $H_i(x_1, x_2) = (w_i d(x, v_i))^2$, we obtain an instance of the above unifying model. But for arbitrary values of $\{s_i\}_i$ some modification is needed. Specifically, this Euclidean 1-center problem can be written as the following variant of the above parametric quadratic model,

$$\begin{aligned} \min \quad & z, \\ \text{s.t.} \quad & H_i(x_1, x_2) \le (z - s_i)^2, \quad \forall i = 1, \dots, n, \\ & (x_1, x_2) \in R_0, z \ge \max_{i=1,\dots,n} s_i. \end{aligned}$$

More generally, consider a Euclidean 1-center problem, where for each customer the transportation cost function is either linear or pure quadratic in the distance to the server. (See [23, 24], for planar 1-center problems with more general nonlinear cost functions.)

Specifically, suppose that $Q_i(d(x, v_i)) = w_i d(x, v_i) + s_i$ for $i = 1, \dots, n'$, and $Q_i(d(x, v_i)) = (w_i d(x, v_i))^2 + s_i$ for $i = n' + 1, \dots, n$. Setting $H_i(x_1, x_2) = (w_i d(x, v_i))^2$, the respective 1-center problem, where we wish to minimize $\max_{i=1,\dots,n} Q_i(d(x, v_i))$, is equivalent to

$$\begin{aligned} \min \quad & z, \\ \text{s.t.} \quad & H_i(x_1, x_2) \le (z - s_i)^2, \quad \forall i = 1, \dots, n', \\ & H_i(x_1, x_2) \le (z - s_i), \quad \forall i = n' + 1, \dots, n, \\ & (x_1, x_2) \in R_0, z \ge \max_{i=1,\dots,n} s_i. \end{aligned}$$

(See Sect. 6.1 for a geometric interpretation of this model.) The above examples suggest the following more general parametric quadratic model,

$$\begin{aligned} \min \quad & z, \\ \text{s.t.} \quad & H_i(x_1, x_2) \le h_i(z), \quad \forall i = 1, \dots, n, \\ & (x_1, x_2) \in R_0, \end{aligned}$$

where the functions $\{h_i(z)\}_i$ are increasing in z for some relevant interval $[a, \infty)$, $a \in \mathbb{R}$. Indeed, we will show in Sect. 6.1 that several generalizations of the classical geometric Euclidean problem can be solved in $O(n)$ time, when the functions $\{h_i(z)\}_i$ are polynomials of bounded degree. Nevertheless, even the latter model is not general enough to accommodate the well known Euclidean 'round-trip' 1-center problems, (see [9, 16] and Sect. 6.5).

2.1 The Extended Parametric Problem

To accommodate for the above and other models, where the transportation cost functions can be transformed to a quadratic formulation, we will consider the following extended model: For $i = 1, \dots, n$,

1. $H_i(x_1, x_2 : z)$ is a continuous piecewise polynomial, of bounded constant degree, of its three variables $(x_1, x_2 : z)$. The subdomains of its pieces are induced by a constant number of hyperplanes.
2. For any real z, $H_i(x_1, x_2 : z)$ is a continuous, convex and piecewise quadratic function of (x_1, x_2).
3. For $z \le z'$, $\{(x_1, x_2) : H_i(x_1, x_2 : z) \le 0\} \subseteq \{(x_1, x_2) : H_i(x_1, x_2 : z') \le 0\}$.

The continuous optimization model that we consider is then,

$$\begin{aligned} \min \quad & z, \\ \text{s.t.} \quad & H_i(x_1, x_2 : z) \le 0, \quad \forall i = 1, \dots, n, \\ & (x_1, x_2) \in R_0. \end{aligned}$$

Similarly, the discrete problem is defined by

$$\begin{aligned} \min \quad & z, \\ \text{s.t.} \quad & H_i(x_1, x_2 : z) \le 0, \quad \forall i = 1, \ldots, n, \\ & (x_1, x_2) \in V = \{v_j = (A_j, B_j) : j = 1, \ldots, m\}. \end{aligned}$$

Unlike the parametric models presented earlier, the above extended parametric model does not require separability of the variable (parameter) z from the location variables (x_1, x_2). This is essential in order to deal with transportation cost functions expressed as sums of distances, and transform them into polynomials. For example, consider the following Euclidean model with the set of planar demand points S, $|S| = n$. When a demand point (customer) $y \in S$ places a call to the server, located at $x = (x_1, x_2)$, the server will travel to y, pick up a package, deliver it to some prespecified destination x_y and return to its home base at x. The transportation cost function of y is given by the weighted tour length of the server, i.e., $Q'_y(x_1, x_2) = w_y(d(x, y) + d(y, x_y) + d(x_y, x))$. This function is the sum of Euclidean distances and it is not a polynomial in x. Nevertheless, as shown in Sect. 6.5, the respective 1-center problem, defined by $\min_x \max_{y \in S} Q'_y(x)$, can be converted into the above extended parametric model.

We note that the functions $H_i(x_1, x_2 : z)$, $i = 1, \ldots, n$, are not assumed to be convex in z. Therefore, the continuous version is not a convex programming problem. Nevertheless, the above extended continuous model constitutes a special case of the parametrized Helly systems discussed in [1, 2, 26, 27]. Specifically, condition 2 implies that the intersection of the parametrized system with the hyperplane $\{z = constant\}$ is a family of convex sets, for which the classical Helly's theorem applies. Condition 3 implies that we have an indexed nested set system. See Sect. 4.2 for the algorithmic implications.

To simplify the presentation and the technical discussion, unless otherwise stated, we will discuss mainly the above simpler linear parametric case, where $H_i(x_1, x_2 : z) = H_i(x_1, x_2) - z$, for $i = 1, \ldots, n$. In Sect. 4.2 we will consider the extended parametric version.

3 Solving the Continuous Planar Piecewise Quadratic Problem

In this section we deal only with the optimization of the linear parametric problem, i.e., minimization of the function $U(x_1, x_2)$, defined in (2).

We first note that the general conic quadratic programming methods, (see [4]), are applicable to some instances of our continuous model. Specifically, when each function H_i is a pure quadratic, i.e., has only one quadratic 'piece', we can directly use conic quadratic programming to solve the models. However, this general approach, leads to polynomial but not strongly polynomial algorithms, and does not yield the linear and subquadratic complexity bounds that we achieve for our bi-variate planar problems.

In the continuous case, the model defined in (2) reduces to a 3-variable convex program with a linear objective function, and $O(n)$ convex, piecewise quadratic constraints. It is a special case of the convex programming model studied by Amenta [1, 2]. Therefore, (from Theorems 5.4.1, 6.2.1 and 6.4.2 in [1]), we conclude that the continuous planar model can be solved as an LP-type problem of dimension 3.

The results by Amenta apply even to the following convex programming problem, defined on $\mathbb{R}^d$, for any fixed d:

$$\begin{aligned} \min \quad & x_1, \\ \text{s.t.} \quad & P_i(x_1, \dots, x_d) \le 0, \qquad \forall i = 1, \dots, n, \\ & (x_1, \dots., x_d) \in \mathbb{R}^d. \end{aligned}$$

$P_i(x_1, \dots, x_d)$, $i = 1, \dots, n$, is a continuous, convex, piecewise polynomial function. Its pieces are defined by a constant number of hyperplanes in $\mathbb{R}^d$. Moreover, assume that the degrees of all pieces of all the functions $\{P_i(x_1, \dots, x_d)\}_i$ are bounded above by some given constant, deg.

All the constraints are algebraic. Therefore the violation and basis computation primitives in [1, 2] can indeed be calculated in constant time if we use the general model of algebraic computation and comparisons over the reals, which is the most common computational model used in computational geometry [18, 38, 39, 43]. Thus, we conclude that for any fixed d, the above continuous piecewise polynomial convex model in $\mathbb{R}^d$, can be solved as an LP-type problem, by the randomized algorithms of Clarkson [12] and Matoušek, Sharir and Welzl [29] in $O(n)$ time.

It can also be solved in $O(n)$ time by the deterministic algorithm of Chazelle and Matoušek [10], which is a derandomization of the randomized algorithm in [29].

We note in passing that the applicability of the framework by Chazelle and Matoušek to the above program follows from the fact that the Vapnik-Chervonenkis dimension (VC-dimension) of the respective range spaces induced by the above convex program is finite. In the case of convex programming testing the finiteness of the dimension reduces to verifying that the arrangement formed by any set of k constraints has $O(k^c)$ cells for some constant c. This clearly holds when the constraints are expressed by fixed-degree polynomials in fixed dimensions, (see Appendix 7.2 in [41]). Since the arrangement can be constructed in near $O(k^c)$ time [3], the computational subsystem oracle requirement in [10], is also satisfied. (If each constraint is expressed by a piecewise fixed-degree polynomial, where the number of pieces is bounded by some constant q, we still have a finite VC-dimension, since the respective arrangement has $O((qk)^c)$ cells.)

We now show that in the case of our planar location problem (2), we do not need to use the general and weak model of algebraic computation and comparisons over the reals. We can instead apply a stronger computational model. The violation and basis computation primitives can be executed in constant time in the planar case if we can obtain an "explicit" representation of intersection points of pairs of quadratic constraints. Specifically, for these two primitives we need to find (x_1, x_2, z) satisfying a system of the form $H_t(x_1, x_2) - z = 0$, where $t = i, j, k$, for some triplet of indices $1 \le i, j, k \le n$. The latter is equivalent to finding (x_1, x_2) satisfying the two (piecewise) quadratic constraints, $H_i(x_1, x_2) - H_j(x_1, x_2) = 0$ and $H_i(x_1, x_2) - H_k(x_1, x_2) = 0$. We need to deal only with nondegenerate systems, corresponding to isolated intersection points (solutions). A pair of planar quadratic constraints has at most four isolated intersection points. A component of any such point is a solution of some univariate quartic polynomial, and therefore can be explicitly represented by radicals. Hence, comparisons over the reals can, in the planar quadratic case, be transformed in constant time into comparisons over the rationals.

We summarize the above with the following theorem.

Theorem 3.1 *The continuous planar piecewise quadratic* 1*-center problem* (2) *can be solved deterministically by an* $O(n)$ (*strongly*) *polynomial algorithm.*

Finally, we note that the results in [10, 12, 29] apply to a more general class of algebraic convex programs. For example, in our context, we can allow each function $H_i(x_1, x_2)$ to represent a positively weighted sum of the Euclidean distances from a server located at (x_1, x_2) to a pair of planar points, say $\{v_i, u_i\}$. This will capture the continuous center round-trip model, (see Sect. 6), and imply its solvability in linear time.

3.1 The Separable Case

In the separable case, i.e., when $H_i(x_1, x_2) = F_i(x_1) + G_i(x_2)$, $i = 1, \dots, n$, is convex separable piecewise quadratic, there is another solution approach, which is not based on the derandomization algorithm of Chazelle and Matoušek [10].

First, we show that by solving $O(\log n)$ univariate subproblems, in $O(n \log n)$ total time, we can identify a convex polygon R^* containing an optimal solution, such that each function $H_i(x_1, x_2)$, $i = 1, \dots, n$, is pure quadratic and separable over R^*. (For this step we do not need to assume that the functions $\{H_i(x_1, x_2)\}_i$ are separable.)

We make the following general position assumption on the set of $\sum_{i=1}^{n} p_i$ partitioning lines. There is a constant c, such that no subset of cardinality greater than c of partitioning lines, can intersect at a point in the plane.

The restriction of $U(x_1, x_2)$ to a line L in the plane is a convex piecewise quadratic function. If we set $p = \max_{i=1,\dots,n} p_i$, the restriction of each function $H_i(x_1, x_2)$ to L has at most $p + 1$ quadratic arcs. Moreover, each pair $H_i(x_1, x_2)$ and $H_j(x_1, x_2)$ of restricted functions intersects at most $s = 4(2p + 1)$ times on L. (This follows from the results on upper envelopes discussed in the next subsection.) Hence, from Theorem 8.1 in the Appendix, the restriction of $U(x_1, x_2)$ to L has $O(\lambda_s(n))$ breakpoints, for $s = 4(2p + 1)$. Nevertheless, (x_1^L, x_2^L), the minimum point of $U(x_1, x_2)$ on L can be directly computed in $O(n)$ time by the algorithm of Zemel [45]. Also, by computing a constant number of directional derivatives of $U(x_1, x_2)$ at (x_1^L, x_2^L) we can determine in $O(n)$ time the halfplane determined by L, which contains a global minimum point of $U(x_1, x_2)$ in the plane. Indeed, let ℓ be the direction of L, and let N_L be the normal vector to L. Let $\mathcal{L}$ denote the set of partitioning lines containing (x_1^L, x_2^L). For each line $L_j \in \mathcal{L}$, let ℓ_j be the direction of L_j. (There are at most c such lines.) The derivatives of $U(x_1, x_2)$ at (x_1^L, x_2^L) in the directions ℓ and $-\ell$ are nonnegative. Evaluate the derivatives of $U(x_1, x_2)$ at (x_1^L, x_2^L) in all the directions in the set $\{N_L, -N_L\} \cup \{\ell_j, -\ell_j : L_j \in \mathcal{L}\}$. By convexity, if all these directional derivatives are nonnegative, then (x_1^L, x_2^L) is a global minimum point of $U(x_1, x_2)$ in the plane. Otherwise, the global minimum is in the unique halfplane (among the pair of halfplanes determined by L), which contains all the negative directional derivatives of $U(x_1, x_2)$ in the above set. (See Sect. 27, [40].)

With the above machinery, we can now use the multidimensional search procedures in [17, 30, 32]. With these methods, applied to the set of $\sum_{i=1}^{n} p_i$ partitioning lines, after solving $O(\log n)$ line restricted subproblems in $O(n \log n)$ total time, we can identify a convex polygon R^*, containing an optimal solution, over which all the piecewise quadratic functions are actually quadratic.

(We note in passing that in the piecewise quadratic model of Foul [22], all the partitioning lines are vertical or horizontal. We can directly perform binary searches over the set of vertical lines and then over the set of horizontal lines, spending $O(n)$ time for each such line. Hence, for this case in $O(n \log n)$ total time we identify a rectangle R^*, containing the optimal solution, over which each function $H_i(x_1, x_2)$ is quadratic.)

$U(x_1, x_2)$ is then the upper envelope of convex quadratics over R^*. Without loss of generality we assume that $R^* \subseteq R_0$.

Turning back to the separable case, the problem of minimizing $U(x_1, x_2)$ over R^* can now be formulated as a 5-variable convex minimization problem where the objective function and all the constraints but two are linear [18]. For $i = 1, \dots, n$, let $F_i^{R^*}(x_1) = a_i x_1^2 + b_i x_1 + c_i$, and $G_i^{R^*}(x_2) = d_i x_2^2 + e_i x_2 + f_i$, denote the quadratic restrictions to R^* of $F_i(x_1)$ and $G_i(x_2)$, respectively. Note that $a_i \ge 0$ and $d_i \ge 0$, for $i = 1, \dots, n$.

$$
\begin{aligned}
\min \quad & z, \\
\text{s.t.} \quad & z \ge a_i v + b_i x_1 + c_i + d_i w + e_i x_2 + f_i, \qquad \forall i = 1, \dots, n, \\
& v \ge x_1^2, \\
& w \ge x_2^2, \\
& x \in R^*.
\end{aligned}
$$

We can then directly apply the algorithm in [18] to locate an optimal solution of $U(x_1, x_2)$ in R^* in $O(n)$ time. (See the Appendix for a short description of Dyer's model.)

The probabilistic center problem in [22], described in the Introduction, is a special case of the separable model.

4 Solving the Discrete Planar Piecewise Quadratic Problem

To solve the general discrete model, we will first need several results from [41] on complexity and algorithms related to envelopes of general planar Jordan arcs and curves. All the relevant material is listed in the Appendix. For further details the reader is referred to [41].

A Jordan arc is an image of the closed unit interval under a continuous bijective mapping, and an unbounded Jordan curve is an image of the open unit interval (or of the entire real line) that separates the plane. Let Γ be a collection of n Jordan arcs in the plane. It is assumed that each pair of arcs intersect in at most s points, for some fixed constant s, and that each arc has at most t points of vertical tangency, for some fixed constant t, so that we can break it into at most $t + 1$ arcs that are monotone in the x_1 direction. In our study all the Jordan arcs are piecewise quadratic. We use the common convention that a (pure) quadratic arc is a connected subset of a conic, which is defined as a set $\{(x_1, x_2) : P(x_1, x_2) = 0\}$, for some, not identically equal to zero, quadratic function $P(x_1, x_2)$.

Consider a pair of x_1-monotone piecewise quadratic Jordan arcs, γ_1, γ_2, each having at most k quadratic pieces. If both γ_1 and γ_2 are pure quadratics, i.e., $k = 1$, then they intersect at most $s = 4$ times. (See Bezout's Theorem, Chap. 16 in [5], which implies that the maximum number of intersections of two conics is 4.)

In general, for any k, the number of intersection points is at most $4(2k - 1)$. To validate the latter bound we note that the total number of breakpoints between the pieces of γ_1 and γ_2 is at most $2k - 2$. Hence, the real line is partitioned into $2k - 1$ adjacent intervals, such that the restrictions of both γ_1 and γ_2 to a given interval are pure quadratic. In particular, γ_1 and γ_2 intersect at most 4 times over any given interval. Therefore, the total number of intersection points of γ_1 and γ_2 is at most $4(2k - 1)$.

Remark 4.1 We view each x_1-monotone Jordan arc as a function of x_1, defined over some closed interval of the real line. If all the arcs in a given collection of x_1-monotone functions have the same common domain, we can clearly extend them to become unbounded x_1-monotone Jordan curves by augmenting horizontal rays at their endpoints. This process will not increase s, the maximum number of intersection points between pairs of curves.

For a real z and $i = 1, \ldots, n$, the convex level set $R_i(z) = \{(x_1, x_2) \in R_0 : H_i(x_1, x_2) \leq z\}$ is assumed to be compact. The arrangement of the p_i lines induces a partition of the plane into $O(p_i^2)$ polyhedral faces. The boundary of $R_i(z)$ is quadratic over each face that it intersects. Due to the convexity of $R_i(z)$, each line is intersected by the boundary of $R_i(z)$ at most twice, and the total number of such intersection points is at most $2p_i$. Therefore, the boundary of $R_i(z)$ can have at most $k_i = 2p_i + 1$ quadratic pieces (arcs). Moreover, this boundary can be decomposed into the union of a concave upper hull and a convex lower hull, both having a total of at most k_i quadratic arcs. The boundary of $R_i(z)$ can contain at most two vertical segments. For convenience, such vertical segments will be attached to the lower hull of the boundary. With the exception of the two vertical segments, all the arcs are monotone in the x_1 direction. Hence, each such arc can be viewed as a function defined over some interval of the x_1 axis. (Some of the pieces can be linear.) Figure 1 illustrates the above concepts. In this example, there are three partitioning lines. The concave upper hull, and the convex lower hull of the boundary of $R_i(z)$ are depicted by the dashed and dotted curves, respectively. The upper hull has 3 pieces, while the lower hull has 5 pieces, including one vertical segment.

The domain of the maximum function $U(x_1, x_2)$ can then be partitioned by the set of $\sum_{i=1}^{n} p_i = O(n)$ lines into $O(n^2)$ polyhedral sets, so that on each such polyhedral set, $U(x_1, x_2)$ is the upper envelope of n convex quadratic functions.

We now return to the solution of the discrete model.

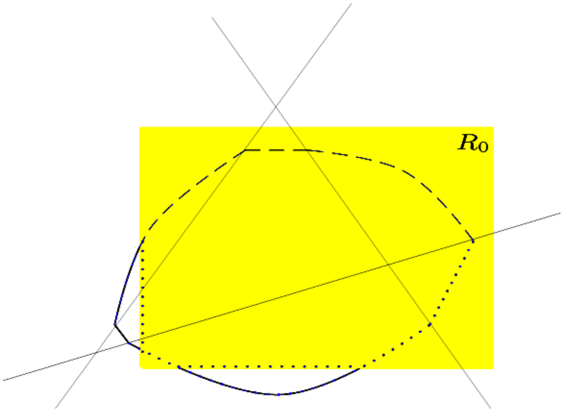

Fig. 1 Illustration of the set $R_i(z)$

Given is a set of m points $V = \{v_j = (A_j, B_j) : j = 1, \dots, m\}$ in the plane. The discrete problem is to compute

$$\min_{j=1,\dots,m} U(A_j, B_j).$$

Using the above notation, in the discrete problem we look for the smallest value of z, such that there is at least one point $v_j \in V$ contained in $R'(z) = \bigcap_{i=1}^{n} R_i(z)$.

We apply the parametric approach of Megiddo [31].

We first need to address the piecewise structure of the functions $\{H_i(x_1, x_2)\}_i$.

As noted above, the boundary of $R_i(z)$ can have at most $k_i = 2p_i + 1$ quadratic arcs. Moreover, this boundary can be decomposed into the union of an x_1-monotone concave upper hull $u_i(z)$ and an x_1-monotone convex lower hull $l_i(z)$, both having a total of at most k_i arcs. We represent each envelope by its $O(k_i)$ quadratic arcs, $\{u_i^k(z)\}_k$ and $\{l_i^k(z)\}_k$, respectively. Let $u(z)$ and $l(z)$ be the lower envelope and the upper envelope of the collections of the concave quadratic arcs $\{u_i^k(z)\}_{ik}$ and the convex quadratic arcs $\{l_i^k(z)\}_{ik}$, respectively. We restrict the domain of the functions $u(z)$ and $l(z)$ to an interval, say $A'(z) \le x_1 \le A''(z)$, which is the intersection of the domains of all the functions (of the variable x_1) $\{u_i(z)\}_i$, $\{l_i(z)\}_i$.

The boundary of the convex feasible set $R'(z)$ is easily obtained from $l(z)$ and $u(z)$ by computing their (at most 2) intersection points in the common domain $A'(z) \le x_1 \le A''(z)$.

Since k_i is assumed to be constant, both $u(z)$ and $l(z)$ are envelopes of $O(n)$ quadratic arcs. We can now apply the results in the Appendix (separately) to the two collections of arcs, with possibly different subdomains of the interval $[A'(z), A''(z)]$, $\{u_i^k(z)\}_{ik}$ and $\{l_i^k(z)\}_{ik}$.

By Theorems 8.2 and 8.4 we conclude that $l(z)$ and $u(z)$ can be generated serially in $O(\lambda_5(n)\log n)$ time, and in $O(\log n)$ parallel time (with $O(\lambda_6(n))$ processors). The complexity of the two envelopes, as well as the boundary of $R'(z)$, is $O(\lambda_6(n))$.

Remark 4.2 There is an alternative way to generate $u(z)$ and $l(z)$. First, convert each quadratic arc function $u_i^k(z)$ and $l_i^k(z)$ to a function defined on the entire interval

$[A'(z), A''(z)]$ by augmenting two tangential lines at its two endpoints. (The convexity/concavity properties are preserved.) Each pair of such functions may intersect at most 6 times. Then, apply Theorems 8.1 and 8.3 for unbounded x_1 monotone families. However, we do not see how to improve upon the complexity achieved by computing envelopes of arcs, as above.

Next, for each point $v_j = (A_j, B_j)$, we test whether v_j is in $R'(z)$. Due to the convexity of $R'(z)$, the concave upper part of the boundary of $R'(z)$ is a subset of the graph of $u(z)$ and its convex lower part is a subset of the graph of $l(z)$. Therefore, by applying a binary search over the breakpoints of $u(z)$ and $l(z)$ it takes $O(\log n)$ time (with a single processor) to determine whether a given point v_j is in $R'(z)$. Hence, it takes $O(\log n)$ time, with $O(m)$ processors, to verify whether $R'(z)$ contains at least one point of V.

To summarize, we have the following lemma.

Lemma 4.1 *For each real z, we can test whether $R'(z)$ contains some point in $V = \{v_1, \dots, v_m\}$, in $O((m + \lambda_5(n)) \log n)$ serial time, or in $O(\log n)$ parallel time (with $O(m + \lambda_6(n))$ processors).*

The optimal solution to the discrete model is given by z^*, the smallest value of z such that $R'(z)$ contains a point in V. With the above machinery we can now find z^* by a direct implementation of the general parametric approach of Megiddo [31]. The total serial running time is then $O((m + \lambda_6(n)) \log^2 n + (m + \lambda_5(n)) \log^3 n)$. (Note that the latter bound is bounded by $O(m \log^3 n + n \log^3 n \log^* n)$[1] [41].)

Theorem 4.1 *The discrete planar piecewise quadratic problem (2) can be solved deterministically in $O((m + \lambda_5(n)) \log^3 n)$ (serial) time.*

For the classical Euclidean and rectilinear geometric problems listed in the Introduction, the implementation of the above approach yields better complexity bounds for solving the discrete cases. Suppose that $m = n$, $V = \{v_i = (A_i, B_i) : i = 1, \dots, n\}$, and $Q_i(d(v_i, x)) = s_i + w_i d(v_i, x)$, for $i = 1, \dots, n$. In the Euclidean case, for each real z, the level sets $\{R_i(z)\}_i$ define a collection of planar discs. The boundaries of each pair of level sets are two circles and obviously they intersect at most $s = 2$ times. Hence, the above parametric approach will solve the discrete problem in this case in $O(n \log^3 n)$ time.

Proposition 4.1 *The discrete planar Euclidean weighted 1-center problem, where the center has to be located at one of the n demand points can be solved deterministically in $O(n \log^3 n)$ time.*

[1] Recall that $\log^* n$ is the minimum number of times q such that q consecutive applications of the log operator will map n to a value smaller than 1, i.e., $\overbrace{\log \dots \log}^{(q)} n \le 1$.

Remark 4.3 The latter bound can possibly be further improved to $O(n\log^2 n)$ by invoking the technique in [13]. The two phases of our parallel algorithm are finding the intersection $R'(z)$ of n planar discs, and testing, for each point in V, whether it belongs to this intersection. Cole's technique requires that the parallel algorithm used will satisfy the bounded fan-in/fan-out property. We note that the second phase of our algorithm uses simultaneous binary searches, and it has this property, (see [13, 14]). On the other hand, we are unaware of any parallel algorithm with the bounded fan-in/fan-out property, to generate the intersection of halfplanes or even congruent planar discs i.e., discs having the same radius.

For comparison purposes, we note that the optimal time to solve the discrete Euclidean unweighted case, where $Q_i(d(v_i,x)) = d(v_i,x)$ for $i = 1,\dots,n$, is $\theta(n\log n)$, (see [28]).

In the rectilinear case, the weighted and the unweighted discrete problems can be solved in $O(n\log n)$ and $O(n)$ times, respectively. First, note that in the planar case, after transforming (A_i,B_i) to $(C_i,D_i) = (A_i + B_i, A_i - B_i)$, $i = 1,\dots,n$, the rectilinear (ℓ_1) distances are replaced by the ℓ_∞ distances. Define $f(x_1) = \max_{i=1,\dots,n} w_i|x_1 - C_i|$ and $g(x_2) = \max_{i=1,\dots,n} w_i|x_2 - D_i|$. The objective value at a planar point (x_1,x_2) is then given by $U(x_1,x_2) = \max\{f(x_1), g(x_2)\}$. Let $C^{min} = \min_{i=1,\dots,n} C_i$, $C^{max} = \max_{i=1,\dots,n} C_i$, $D^{min} = \min_{i=1,\dots,n} D_i$, and $D^{max} = \max_{i=1,\dots,n} D_i$.

In the unweighted model, for $i = 1,\dots,n$, $U(C_i,D_i) = \max\{C_i - C^{min}, C^{max} - C_i, D_i - D^{min}, D^{max} - D_i\}$. The optimal discrete solution can therefore be computed in $O(n)$ time.

In the weighted version, the piecewise linear and convex upper envelopes $f(x_1)$ and $g(x_2)$ can be constructed in $O(n\log n)$ time, by using standard divide and conquer procedures. (See also the results in Theorems 8.1, 8.2 in the Appendix, with $s = 1$.) Then, for $i = 1,\dots,n$, it takes $O(\log n)$ time to compute $f(C_i)$, $g(D_i)$, and $U(C_i,D_i)$. Therefore, the optimal discrete solution for the weighted version is computable in $O(n\log n)$ time. In addition to the above deterministic algorithms there also exists the randomized algorithm in [26, 27] which requires a linear number of operations to solve the planar weighted discrete model. This algorithm is based on a discrete LP-type model developed by the author.

Proposition 4.2 *The discrete planar rectilinear weighted* 1*-center problem, where the center has to be located at one of the n demand points can be solved deterministically in $O(n\log n)$ time. The unweighted version can be solved deterministically in $O(n)$ time.*

4.1 Using the Parametric Approach to Solve the Continuous Model

The parametric approach which we have used above for solving the discrete model, can also be used to solve the continuous models. However, the complexity is subquadratic. We first demonstrate how to apply the approach to the linear parametric problem. In the next section we apply it to the extended parametric model presented in Sect. 2.1. Specifically, in the continuous case, we can first identify the polygon R^*, containing an optimal solution, over which all the functions $H_i(x_1, x_2)$, $i = 1, \ldots, n$, are pure quadratic. As explained above this phase takes $O(n \log n)$ time. We then apply the parametric approach.

We assume that R^* has already been computed and $H_i(x_1, x_2)$ is quadratic for $i = 1, \ldots, n$. For a given real z and $i = 1, \ldots, n$, the convex level set $R_i(z) = \{(x_1, x_2) \in R_0 : H_i(x_1, x_2) \leq z\}$ has been assumed to be compact. We will say that z is feasible if the convex feasible set

$$R(z) = R^* \cap \left(\bigcap_{i=1}^{n} R_i(z) \right),$$

is nonempty. The optimal solution to the above model is given by z^*, the smallest value of z such that $R(z)$ is nonempty.

Thus, given a real z, our goal is to test efficiently whether z is feasible or not. We will do it by constructing the piecewise quadratic boundary of $R(z)$.

As noted above, for each $i = 1, \ldots, n$, the boundary of $R_i(z)$ can be decomposed into the union of an x_1-monotone concave quadratic arc $H_i^+(x_1, x_2) = z$, and an x_1-monotone convex quadratic arc $H_i^-(x_1, x_2) = z$. (We view both $H_i^-(x_1, x_2)$ and $H_i^+(x_1, x_2)$ as functions of the variable x_1.) Moreover, for each pair i, j, the quadratic arcs $H_i^+(x_1, x_2) = z$ and $H_j^+(x_1, x_2) = z$ ($H_i^-(x_1, x_2) = z$ and $H_j^-(x_1, x_2) = z$) intersect at most 4 times.

Let $u(z)$ denote the lower envelope of the collection of the concave quadratic arcs $\{H_i^+(x_1, x_2) = z\}_i$, and let $l(z)$ denote the upper envelope of the collection of the convex quadratic arcs $\{H_i^-(x_1, x_2) = z\}_i$. We view both $u(z)$ and $l(z)$ as functions of x_1, and restrict their domain to an interval, say $A'(z) \leq x_1 \leq A''(z)$, which is the intersection of the domains of all $2n$ arcs (functions) $\{H_i^+(x_1, x_2) = z\}_i$, $\{H_i^-(x_1, x_2) = z\}_i$. Note that $u(z)$ is concave and $l(z)$ is convex over this common domain. The boundary of $R(z)$ is easily obtained from $u(z)$ and $l(z)$ by computing the (at most 2) intersection points between them in the interval $[A'(z), A''(z)]$, and their intersection points with the boundary of the rectangle R^*.

Using the above mentioned properties of the two collections, $\{H_i^+(x_1, x_2) = z\}_i$, $\{H_i^-(x_1, x_2) = z\}_i$, Theorems 8.1 and 8.3 in the Appendix imply that $u(z)$ and $l(z)$ can be generated serially in $O(\lambda_4(n) \log n)$ time, and in $O(\log n)$ parallel time with $O(\lambda_4(n))$ processors. (To see the applicability of these results note that all the arcs in $\{H_i^+(x_1, x_2) = z\}_i$, $\{H_i^-(x_1, x_2) = z\}_i$ are restricted to the same domain, $[A'(z), A''(z)]$. Therefore, from Remark 4.1 in the previous section, each of the two collections can be viewed as a set of n unbounded x_1-monotone Jordan curves with at most 4 intersections between any pair of curves.)

The above complexity bounds include also the effort to generate all the arcs $\{H_i^+(x_1, x_2) = z\}_i$, $\{H_i^-(x_1, x_2) = z\}_i$, as well as the bounds $A'(z)$ and $A''(z)$. The complexity of the two envelopes $u(z)$ and $l(z)$ and the complexity of the boundary of $R(z)$ are all $O(\lambda_4(n))$.

To summarize, for each real z, we can now test whether $R(z)$ is empty or not in $O(\lambda_4(n) \log n)$ serial time or in $O(\log n)$ parallel time (with $O(\lambda_4(n))$ processors).

With the above machinery, we can find z^* by directly implementing the general parametric approach of Megiddo [31]. The total serial running time is $O(\lambda_4(n) \log^3 n)$.

($\lambda_4(n) = \theta(n2^{\alpha(n)})$, where $\alpha(n)$ is the inverse of the Ackermann function, and it satisfies $\lambda_4(n)\log^3 n = o(n\log^3 n \log^* n)$, Chap. 3 in [41].)

4.2 Solving the Extended Parametric Quadratic Model

In this section we briefly discuss the solution of the extended parametric quadratic model,

$$\begin{aligned} \min \quad & z, \\ \text{s.t.} \quad & H_i(x_1, x_2 : z) \le 0, \quad \forall i = 1, \dots, n, \\ & (x_1, x_2) \in R_0, \end{aligned}$$

where each function $H_i(x_1, x_2 : z)$ satisfies the three properties listed in Sect. 2. As noted in Sect. 2, the continuous model is a special case of the parametrized Helly systems discussed in [1, 2]. The three properties ensure that the violation and basis computation primitives can indeed be calculated in constant time. Therefore, our continuous model can be solved in $O(n)$ time by the randomized algorithms described by Amenta. It is not clear whether the derandomization method of Chazelle and Matoušek [10] is applicable to the parametrized Helly systems.

To obtain efficient deterministic algorithms for both the continuous and the discrete versions of this general model, we can apply the same approach used above to solve the linear parametric model. We only need to modify several definitions and address the issue of computing and representing intersection points of certain quadratic arcs.

For a real z and $i = 1, \dots, n$, the convex level set $R_i(z)$ is now defined by $R_i(z) = \{(x_1, x_2) \in R_0 : H_i(x_1, x_2 : z) \le 0\}$. From the properties of the functions $\{H_i\}_i$, it is clear that for any fixed z, the level sets have the same piecewise quadratic structure as the level sets in Sect. 3.1. In particular, the boundary of $R_i(z)$ can be decomposed into the union of a concave upper hull and a convex lower hull, both consisting of a constant number of quadratic arcs. We can then apply the serial and parallel algorithms to construct and represent $R'(z) = \bigcap_{i=1}^{n} R_i(z)$, the intersection of the n level sets.

We also note that $R'(z)$ is monotone in the parameter z, i.e., if $z \le z'$, then $R'(z) \subseteq R'(z')$. Therefore, the parametric algorithm of Megiddo [31] is directly applicable to find the smallest value of the parameter such that $R'(z)$ is either nonempty (in the continuous model), or contains at least one of the points of V (in the discrete version). The only remaining question is the representation of the critical values of the parameter z that we encounter throughout the application of the parametric algorithm.

Suppose that $H'_j(x_1, x_2 : z) = 0$ and $H'_k(x_1, x_2 : z) = 0$ represent two quadratic arcs. Each function $H_i(x_1, x_2 : z)$ is piecewise polynomial in $(x_1, x_2 : z)$ and quadratic in (x_1, x_2). Therefore, each intersection point of the above two quadratic arcs, can be represented explicitly by formulas involving quadratic, cubic and quartic roots of polynomials in z. Each critical value of the parameter z that we encounter is then a root of an equation obtained by equating two such formulas. Hence, any critical value z' is itself a root of some polynomial of the parameter z, which can be expressed explicitly. (See [38, 39, 44] for computational methods to handle algebraic functions and their roots.)

We conclude that the complexity bounds of solving the above parametric models coincide with the respective bounds given in Sect. 3. Specifically, the continuous and the discrete versions are solved in $O(\lambda_5(n)\log^3 n)$ and $O(m\log^3 n + \lambda_5(n)\log^3 n)$ times, respectively. In the continuous case, the bound reduces to $O(\lambda_4(n)\log^3 n)$ if for any real z, and $i = 1, \dots, n$, $H_i(x_1, x_2 : z)$ is a convex (pure) quadratic function of (x_1, x_2).

Theorem 4.2 *The continuous and discrete planar extended parametric piecewise quadratic problems can be solved deterministically in $O(\lambda_5(n)\log^3 n)$ and $O(m\log^3 n + \lambda_5(n)\log^3 n)$ (serial) times, respectively. The continuous version can also be solved as an LP-type problem by a randomized algorithm in $O(n)$ time.*

The above scheme is not recommended as a practical tool. Its main purpose is to prove that several planar 1-center problems, like the Euclidean round-trip problem, where the degree of the piecewise polynomial functions $\{H_i(x_1, x_2 : z)\}_i$ is bounded above by a small constant, (e.g., at most 4), have subquadratic complexity.

5 Extensions to $\mathbb{R}^d$

We note that some of the results in the paper can be extended to $\mathbb{R}^d$, for any fixed d. Consider the extension of the planar continuous model discussed in Sect. 3. In the extended model we assume that all the pieces of the convex and piecewise quadratic functions $\{H_i(x_1, x_2, \dots, x_d)\}_i$ in $\mathbb{R}^d$, are induced by a grid of $\mathbb{R}^d$, defined by a set of $O(n)$ partitioning hyperplanes. First we observe that a subquadratic deterministic $O(n(\log n \log\log n)^{2^d-1})$ algorithm for the continuous model follows directly from the algebraic convex model of Toledo [44]. Secondly, the results of Clarkson [12], Matoušek, Sharir and Welzl [29], Chazelle and Matoušek [10] and Dyer [18] imply the existence of randomized and deterministic linear time algorithms for the respective models in $\mathbb{R}^d$. (See Sect. 3 for a note on the applicability of the derandomization algorithm of Chazelle and Matoušek [10].) We summarize the above with the following propositions.

Proposition 5.1 *For any fixed positive integer d, the continuous piecewise quadratic problem in $\mathbb{R}^d$ can be solved deterministically in $O(n)$ time.*

Proposition 5.2 *For any fixed positive integer d, there is a randomized algorithm to solve the continuous extended parametric piecewise quadratic problem in $\mathbb{R}^d$ in $O(n)$ time.*

Again, we note that the model of computation that we use to solve our center problem is that of algebraic computation and comparisons over the reals. In particular, the algorithms that we discuss output a small system of polynomial equations which determine the optimal solution [18, 38, 39, 44].

For the effectiveness of the above algorithms for relatively large values of d, we refer the reader to the computational studies reported in [46].

It is not clear to us how the parametric approach, and the results on the intersection properties of quadratic arcs, which we applied in Sect. 4 to solve the discrete planar models, can be extended to $\mathbb{R}^d$, for $d \geq 3$.

In the discrete model we are given a set of n points $V = \{v_i = (v_{i1}, \ldots, v_{id}) : i = 1, \ldots, n\}$ in $\mathbb{R}^d$, and the goal is to compute $\min_{i=1,\ldots,n} U(v_{i1}, \ldots, v_{id})$, where $U(x_1, \ldots, x_d) = \max_{i=1,\ldots,n} H_i(x_1, \ldots, x_d)$. Note that for a fixed d, the discrete model can be solved in quadratic time by computing the objective at each point of V.

With the exception of some special cases that we discuss below, we are unaware of any subquadratic algorithm which solves the discrete 1-center problem in $\mathbb{R}^d$, for any fixed $d \geq 3$.

In particular, we do not yet know how to solve the discrete weighted or unweighted Euclidean problem in $\mathbb{R}^d$ for a fixed $d \geq 3$ in subquadratic time.

For comparison purposes we note that the discrete weighted (unweighted) rectilinear 1-center problem in $\mathbb{R}^d$ can be solved deterministically in $O(n \log n)$ ($O(n)$) time, for any fixed d. This latter result follows directly from the formulation in [35, Sect. 2.2, p. 119], which is used there to show the solvability of the continuous weighted 1-center problem in $\mathbb{R}^d$, for any fixed d, in $O(n)$ time. This formulation is based on the simple observation that for any $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$, the rectilinear norm, $d_1(y, 0)$, can be represented as the maximum of absolute values of 2^d linear functions of the form $\delta_1 y_1 + \cdots + \delta_d y_d$, where $\delta_j \in \{-1, 1\}$. As a result, to compute the objective value of the 1-center rectilinear problem for query points, it is sufficient to generate $O(2^d)$ 1-dimensional upper envelopes. Each such envelope is the pointwise maximum function of n single variable linear functions, and it can be computed in $O(n \log n)$ and $O(n)$ times, for the weighted and unweighted versions, respectively. The total efforts needed to generate all these envelopes are $O(2^d n \log n)$ for the weighted model and $O(2^d n)$ for the unweighted model. Given any query point $v_i \in \mathbb{R}^d$, with this machinery it now takes $O(2^d \log n)$ ($O(2^d)$) time to compute the objective value at v_i for the weighted (unweighted) version. To solve the discrete rectilinear model we compute the objective at all points in $V = \{v_1, \ldots, v_n\}$. The total efforts to solve the discrete rectilinear problem are then $O(2^d n \log n)$ and $O(2^d n)$ for the weighted and unweighted versions, respectively.

We summarize the above with the following proposition.

Proposition 5.3 *For any fixed positive integer d, the discrete weighted (unweighted) rectilinear problem in $\mathbb{R}^d$ can be solved deterministically in $O(n \log n)$ ($O(n)$) time.*

Finally, we cite the $O(n)$ discrete LP-type randomized algorithm of Halman [26, 27], mentioned above, to solve the discrete weighted ℓ_∞ 1-center problem in $\mathbb{R}^d$ for any fixed $d \geq 3$. We note that this algorithm can also be used to solve the rectilinear, i.e., ℓ_1 1-center model. It is not yet known whether this randomized algorithm can be derandomized.

6 Applications

In addition to the single facility location problems which have motivated our study, and are described in the Introduction, in this section we present several additional problems that can be solved using the general unifying model presented above.

6.1 Formulating and Solving the Euclidean Weighted 1-Center Problem and Some Variants in $\mathbb{R}^d$

Suppose that $\{s_i\}_{i=1,\dots,n}$ are arbitrary reals and $\{w_i\}_{i=1,\dots,n}$ are nonnegative reals. Consider the Euclidean weighted 1-center problem with addends,

$$\begin{aligned} \min \quad & z, \\ \text{s.t.} \quad & s_i + w_i d(v_i, x) \le z, \quad \forall i = 1, \dots, n, \\ & x \in \mathbb{R}^d. \end{aligned} \tag{3}$$

Define $s = \max_{i=1,\dots,n} s_i$. Note that $z \ge s$. Then, substitute $z = z' + s$ to reduce the problem to,

$$\begin{aligned} \min \quad & (z' + s), \\ \text{s.t.} \quad & (w_i d(v_i, x))^2 \le (z' + s - s_i)^2, \quad \forall i = 1, \dots, n, \\ & z' \ge 0, \\ & x \in \mathbb{R}^d. \end{aligned}$$

An equivalent formulation is

$$\begin{aligned} \min \quad & z', \\ \text{s.t.} \quad & (w_i d(v_i, x))^2 \le (z')^2 + 2(s - s_i)z' + (s - s_i)^2, \quad \forall i = 1, \dots, n, \\ & z' \ge 0, \\ & x \in \mathbb{R}^d. \end{aligned}$$

From the definition of s, we note that the above model is a special case of the following *parametric separable quadratic model*:

$$\begin{aligned} \min \quad & z, \\ \text{s.t.} \quad & x^T D_i x + c_i^T x + b_i \le h_i(z), \quad \forall i = 1, \dots, n, \\ & z \ge 0, \\ & x \in \mathbb{R}^d, \end{aligned} \tag{4}$$

where for each $i = 1, \dots, n$, D_i is a nonnegative diagonal matrix, $c_i \in \mathbb{R}^d$, $b_i \in \mathbb{R}$, and $h_i(z) = \sum_{j=1}^{m_i} b_{ij} z^j$ is a polynomial with nonnegative real coefficients. For $i = 1, \dots, n$, let $d_i \in \mathbb{R}^d$ denote the diagonal vector of the diagonal matrix D_i. Let $m = \max_{i=1,\dots,n} m_i$.

Finally, replacing the objective function z by z^m, substituting $u = z^m$, $v_j = -z^j = -(u)^{j/m}$, $j = 1, \ldots, m$, and $y_k = x_k^2$, $k = 1, \ldots, d$, we obtain the following equivalent problem,

$$\begin{aligned}
\min \quad & u, \\
\text{s.t.} \quad & y^T d_i + x^T c_i + b_i + \sum_{j=1}^{m_i} b_{ij} v_j \le 0, \quad \forall i = 1, \ldots, n, \\
& -(u)^{j/m} \le v_j, \quad \forall j = 1, \ldots, m, \\
& x_k^2 \le y_k, \quad \forall k = 1, \ldots, d, \\
& u \ge 0, \\
& x, y \in \mathbb{R}^d.
\end{aligned}$$

We note that the $d + m$ nonlinear constraints in the above problem are convex since $u \ge 0$.

Summarizing, we obtain the following result by applying the algorithm in [18].

Proposition 6.1 *Assuming that d and m are constant, the parametric separable quadratic problem* (4) *is solvable deterministically in $O(n)$ time.*

Staying within the framework of 1-center problems with general nonlinear cost functions, (see [23, 24]), we now use the above parametric separable quadratic model to identify additional classes of nonlinear problems that are solvable in linear time. Consider the following extension of the Euclidean 1-center problem, where for each $i = 1, \ldots, n$, $Q_i(d(x, v_i)) = (w_i d(x, v_i))^{t_i} + s_i$. (If $t_i = 1$ for $i = 1, \ldots, n$, we obtain the Euclidean 1-center problem with addends discussed above.)

Setting $s = \max_{i=1,\ldots,n} s_i$ the respective center problem can be formulated as

$$\begin{aligned}
\min \quad & z', \\
\text{s.t.} \quad & (w_i d(v_i, x))^2 \le (z' + (s - s_i))^{2/t_i}, \quad \forall i = 1, \ldots, n, \\
& z' \ge 0, \\
& x \in \mathbb{R}^d.
\end{aligned} \tag{5}$$

Consider first the following class of concave cost functions of the service distance. Let m be a constant positive integer. Suppose that for each $i = 1, \ldots, n$, the transportation cost function $Q_i(d(v_i, x))$ is the sum of the addend s_i and the m_i-th root of $(w_i d(v_i, x))^2$, for some positive integer $m_i \le m$. Formally, in terms of the above model $t_i = 2/m_i$, for $i = 1, \ldots, n$. We refer to this model as the *general* 1*-center center problem with addends*.

If all the addends $\{s_i\}_i$ are zero, the linear solvability holds for a wider class of cost functions. Suppose that for each $i = 1, \ldots, n$, $t_i \in T'$, where $T' = \{f_1, \ldots, f_m\}$ is a finite set of constant cardinality of positive real numbers. (Note that Q_i is convex (concave) if $t_i = f_k$ and $f_k \ge 1$ ($t_i = f_k \le 1$).)

Let $f = \min_{j=1,\ldots,m} f_j$. The respective center problem is then

$$
\begin{aligned}
\min \quad & z^{2/f}, \\
\text{s.t.} \quad & (w_i d(v_i, x))^2 \le (z)^{2/t_i}, \quad \forall i = 1, \ldots, n, \\
& z \ge 0, \\
& x \in \mathbb{R}^d.
\end{aligned}
\tag{6}
$$

We refer to this model as the *extended* 1*-center center problem without addends*. Finally, define $u = z^{2/f}$, and $v_j = -z^{2/f_j} = -u^{f/f_j}$, for $j = 1, \ldots, m$, to obtain the following equivalent model with m nonlinear convex constraints. (We use the convention that $v_{i(j)} = v_k$ if and only if $t_i = f_k$.)

$$
\begin{aligned}
\min \quad & u, \\
\text{s.t.} \quad & (w_i d(v_i, x))^2 + v_{i(j)} \le 0, \quad \forall i = 1, \ldots, n, \\
& u \ge 0, \\
& -u^{f/f_j} \le v_j, \quad \forall j = 1, \ldots, m, \\
& x \in \mathbb{R}^d.
\end{aligned}
$$

Summarizing the discussion in this section we present the next proposition.

Proposition 6.2 *The general* 1*-center problem with addends* (5) *and the extended* 1*-center problem without addends* (6) *are both solvable deterministically in* $O(n)$ *time.*

6.2 A Minimax Location Problem with Respect to a Fixed Number of Demand Regions

The second model is based on [8] and [34]. In this location problem there is a fixed number, say r, of compact regions $\{A_1, \ldots, A_r\}$ in $\mathbb{R}^2$, representing communities of customers. A new facility (server), modeled by a point in $\mathbb{R}^2$ needs to be located in the plane. There are n measures (characteristics), modeled respectively by n distance functions $\{d_1, \ldots, d_n\}$, which affect the location of the new facility. Specifically, if x is the location of the facility, the service distance of region A_j according to the n characteristics is given by

$$
D_j(x) = \min_{y \in A_j} \max_{i=1,\ldots,n} w_i d_i(x, y).
$$

The *unconstrained single facility doubly weighted minimax location problem* takes the following form:

$$
\min_{x \in \mathbb{R}^2} [V(x) = \max_{j=1,\ldots,n} \{u_j D_j(x)\}], \tag{7}
$$

where w_i is a positive weight associated with the i-th characteristic, $i = 1, \ldots, n$; and u_j the positive weight of A_j, $j = 1, \ldots, r$. (For example, w_i may signify the relative importance of the i-th characteristic, while u_j may represent the size of the population of region A_j.) Thus, if each set A_j, $j = 1, \ldots, r$, is assumed to be closed convex and polygonal, Problem (7) may be formulated as the following mathematical problem:

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & \bar{w}_{ij} d_i(x, y_j) \le z, \quad i = 1, \dots, n,\ j = 1, \dots, r, \\ & y_j \in A_j, \quad j = 1, \dots, r, \\ & x \in \mathbb{R}^2, \end{aligned} \tag{8}$$

where $\bar{w}_{ij} = w_i u_j$, $\forall i, j$. Assume further that each set A_j, $j = 1, \dots, r$, has a fixed number of edges, and each distance measure d_i, $i = 1, \dots, n$, is induced by some polyhedral planar norm with a fixed number of faces, e.g., rectilinear.

Proposition 6.3 *The planar unconstrained single facility doubly weighted minimax location problem with n polyhedral norms is solvable deterministically in $O(n)$ time.*

Proof The problem reduces to a linear program in fixed dimension, (there are $2r + 3$ variables). Hence, it can be solved in $O(n)$ time by the algorithm in [32]. □

We note that the paper by Brimberg and Wesolowsky [8] provides no complexity analysis of the above model, while the complexity reported in [34] is superquadratic.

Finally, we note that even the case where a fixed number of the characteristics are measured by weighted Euclidean distances can also be solved in linear time. Suppose that a fixed number, say m, of the d_i are Euclidean distances, while the rest $n - m$ are rectilinear. The respective problem can be formulated as

$$\begin{aligned} \min \quad & \max\{v, \sqrt{w}\} \\ \text{s.t.} \quad & \bar{w}_{ij}(y_{j1} - x_1) + \bar{w}_{ij}(y_{j2} - x_2) \le v, \quad \forall i = m+1, \dots, n,\ j = 1, \dots, r, \\ & -\bar{w}_{ij}(y_{j1} - x_1) + \bar{w}_{ij}(y_{j2} - x_2) \le v, \quad \forall i = m+1, \dots, n,\ j = 1, \dots, r, \\ & \bar{w}_{ij}(y_{j1} - x_1) - \bar{w}_{ij}(y_{j2} - x_2) \le v, \quad \forall i = m+1, \dots, n,\ j = 1, \dots, r, \\ & -\bar{w}_{ij}(y_{j1} - x_1) - \bar{w}_{ij}(y_{j2} - x_2) \le v, \quad \forall i = m+1, \dots, n,\ j = 1, \dots, r, \\ & \bar{w}_{ij}^2((x_1 - y_{j1})^2 + (x_2 - y_{j2})^2) \le w, \quad \forall i = 1, \dots, m,\ j = 1, \dots, r, \\ & y_j \in A_j, \quad j = 1, \dots, r, \\ & x \in \mathbb{R}^2. \end{aligned} \tag{9}$$

The above can be rewritten in a form that fits the model in [18]:

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & v^2 \le z, \\ & w \le z, \\ & \bar{w}_{ij}(y_{j1} - x_1) + \bar{w}_{ij}(y_{j2} - x_2) \le v, \quad \forall i = m+1, \dots, n,\ j = 1, \dots, r, \end{aligned}$$

$$-\bar{w}_{ij}(y_{j1}-x_1)+\bar{w}_{ij}(y_{j2}-x_2)\le v,\quad \forall i=m+1,\dots,n,\ j=1,\dots,r,$$
$$\bar{w}_{ij}(y_{j1}-x_1)-\bar{w}_{ij}(y_{j2}-x_2)\le v,\quad \forall i=m+1,\dots,n,\ j=1,\dots,r,$$
$$-\bar{w}_{ij}(y_{j1}-x_1)-\bar{w}_{ij}(y_{j2}-x_2)\le v,\quad \forall i=m+1,\dots,n,\ j=1,\dots,r,$$
$$\bar{w}_{ij}^2((x_1-y_{j1})^2+(x_2-y_{j2})^2)\le w,\quad \forall i=1,\dots,m,\ j=1,\dots,r,$$
$$y_j\in A_j,\quad j=1,\dots,r,$$
$$x\in\mathbb{R}^2.$$

The above reduction leads to the following result.

Proposition 6.4 *The planar unconstrained single facility doubly weighted minimax location problem with n weighted rectilinear norms, and m weighted Euclidean norms is solvable deterministically in $O(n)$ time, when m is constant.*

6.3 Location Problems with Respect to Several Scenarios

The problem studied in this section was first introduced in Fernández et al. [21] who considered a location problem with respect to several scenarios. That paper does not discuss complexity issues of the problem and it only suggests that the problem can be solved using a general non linear programming approach. Let A be a finite set, ($|A|=n$), of existing facilities in $\mathbb{R}^2$, and let W be a finite set, ($|W|=m$), of weight vectors $w\in\mathbb{R}^n$. Each $w\in W$ satisfies $\sum_{a\in A} w_a=1$ and $w\ge 0$; $\forall a\in A$.

Each weight $w\in W$ represents a location scenario while w_a is the probability given to the existing facility $a\in A$ in the scenario w. It is assumed that costs are measured by the squared Euclidean norm, $\|\cdot\|_2^2$. Examples of such a quadratic formulation are the problems of locating hospitals, fire stations, police stations, and other emergency service agencies. (See e.g., [21, 36].) Therefore, the minimax regret problem is

$$\min_{x\in\mathbb{R}^2}\max_{w\in W}\left(\sum_{a\in A} w_a\|x-a\|_2^2-\sum_{a\in A} w_a\|x(w)-a\|_2^2\right) \tag{10}$$

where $x(w)$ is the optimal solution of problem:

$$\min_{x\in\mathbb{R}^2}\sum_{a\in A} w_a\|x-a\|_2^2.$$

It is well-known that $x(w)=\frac{\sum_{a\in A} w_a a}{\sum_{a\in A} w_a}=\sum_{a\in A} w_a a$. Hence, we reformulate Problem (10) as

$$\min_{x\in\mathbb{R}^2}\max_{w\in W}\left(\sum_{a\in A} w_a\|x-a\|_2^2-\sum_{a\in A} w_a\Big\|\sum_{a\in A} w_a a-a\Big\|_2^2\right).$$

Expanding the scalar products in the above expression and simplifying, we obtain the following equivalent formulation. (See [21] for the details.)

$$\min_{x\in\mathbb{R}^2}\max_{w\in W}\|x-x(w)\|_2^2.$$

This problem is just the Euclidean unweighted center problem with respect to the points $\{x(w) : w \in W\}$,

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & (x_1 - x_1(w))^2 + (x_2 - x_2(w))^2 \le z, \quad \forall w \in W, \end{aligned}$$

and it can be solved in $O(m)$ time by Megiddo [30]. (This effort does not include the $O(mn)$ time to compute the set $\{x(w) : w \in W\}$.)

6.4 The Scalarized and Constrained Version of Ohsawa's Model

This problem is based on the scalarized version of the quadratic bicriteria mean-center planar location models of Ohsawa [36]. (See [36] for further details.)

We consider doubly weighted versions of his scalarized and constrained models, and show that even these can be formulated as special cases of the model in [18]. (We use the same notation as in the previous problem.) For $a \in A$, the mean and center nonnegative weights of demand point a are denoted by w_a and w'_a, respectively. M and M' are nonnegative reals that bound the weighted mean and center values.

$$\begin{aligned} \min \quad & \sum_{a \in A} w_a[(a_1 - x_1)^2 + (a_2 - x_2)^2] + \max_{a \in A}(w'_a[(a_1 - x_1)^2 + (a_2 - x_2)^2]), \\ \text{s.t.} \quad & \sum_{a \in A} w_a[(a_1 - x_1)^2 + (a_2 - x_2)^2] \le M, \\ & \max_{a \in A}(w'_a[(a_1 - x_1)^2 + (a_2 - x_2)^2]) \le M'. \end{aligned} \tag{11}$$

Proposition 6.5 *The scalarized doubly weighted version of Ohsawa's planar location model* (11) *is solvable deterministically in linear time.*

Proof The problem is equivalent to the next model that fits Dyer's formulation.

$$\begin{aligned} \min \quad & (v + u), \\ \text{s.t.} \quad & \sum_{a \in A} w_a[(a_1 - x_1)^2 + (a_2 - x_2)^2] \le v, \\ & \sum_{a \in A} w_a[(a_1 - x_1)^2 + (a_2 - x_2)^2] \le M, \\ & w'_a[(a_1 - x_1)^2 + (a_2 - x_2)^2] \le u, \quad \forall a \in A, \\ & w'_a[(a_1 - x_1)^2 + (a_2 - x_2)^2] \le M', \quad \forall a \in A. \end{aligned}$$

Note that all the constraints are convex and separable quadratic. Therefore, this model can be solved in $O(n)$ time as a special case of Dyer's model. □

For comparison purposes, Ohsawa [36] only considers the unweighted version and reports a complexity of $O(n \log n)$.

6.5 Other Problems

In this section we consider two variants of the depot one-way and round-trip location problems discussed in [42].

For the first model, let $A \subset \mathbb{R}^2$, be a finite set, representing the locations of existing depots, and $S \subset \mathbb{R}^2$ be the set of customers. Let $m = |A|$. Suppose that $x \in \mathbb{R}^2$ denotes the location of the single server (center). When a customer $y \in S$ places a call for service, the server at x will travel to y, pick up a package, and deliver the package to the closest depot to y in A. $d(x, y) + d(y, A)$ represents the distance traveled by the server from its home base x to customer y and then to the closest depot to y in A. The problem to be solved is:

$$\min_{x \in \mathbb{R}^2} \max_{y \in S} \{d(x, y) + d(y, A)\}.$$

(We use the common notation $d(y, A) = \min_{u \in A} d(y, u)$.) This is the depot one-way model, where the tour initiates at the home base of the server and terminates at the depot. We show that the above problem can be transformed into a minimax problem with positive addends which in turns fits into our model. First, we note that the variable y can be restricted to a finite subset when S is a polygon. Let V_a denote the a-th cell of the Voronoi diagram of the set A, i.e., $V_a = \{z \in \mathbb{R}^2 : d(z, a) = d(z, A)\}$. For any given x, the inner maximum when y is restricted to $V_a \cap S$ is attained at a corner point of the convex hull of $V_a \cap S$, (due to the convexity of the objective $d(x, y) + d(y, a)$). For example, when S is a planar polygon with $O(k)$ edges, and d is the Euclidean distance function, each edge of S can intersect $O(m)$ Voronoi cells. (Recall that the complexity of the Voronoi diagram is $O(m)$, and the time to construct it is $O(m \log m)$.) Hence, the total number of corner points that we have to consider is $O(mk)$. Let S^* be the set of all the extreme points of the convex hulls of the sets $\{V_a \cap S\}_a$. In this case the model reduces to

$$\min_{x \in \mathbb{R}^2} \max_{y \in S^*} \{d(x, y) + d(y, A)\},$$

which corresponds to the geometric problem of finding a circle of minimum radius enclosing a given set of $O(mk)$ circles. (Each point $y \in S^*$ defines a circle of radius $d(y, A)$.) Hence, from the results in Sect. 6.1, the latter problem is solvable in $O(mk)$ time.

Proposition 6.6 *The depot one-way Euclidean planar single facility center problem defined by a set of depots A, $|A| = m$, and a polygonal customer set S, having $O(k)$ edges, can be solved deterministically in $O(mk + m \log m)$ time.*

We note that a version of the above model when S is a finite set of n points, was solved in [42] in $O((n + m) \log m)$ time.

Next we address the following variant of the 'customer round-trip' planar single facility center problem as introduced in [15] and [6], and later analyzed by Tamir and Halman [42]. In this model we assume that the customer set S is finite, ($|S| = n$), and each customer $y \in S$ is associated with a unique depot, say $x_y \in A$. (For example,

x_y can be the closest depot to y in A.) Suppose that x denotes the location of the single server (center). When a customer $y \in S$ places a call for service, the server at x will travel to y, pick up a package, deliver the package to x_y in A, and return to its home base. $d(x, y) + d(y, x_y) + d(x_y, x)$ represents the total round-trip distance traveled by the server. Each customer $y \in S$ has a positive weight w_y, and the goal is to establish one server x optimizing the following problem:

$$\min_{x \in \mathbb{R}^2} \max_{y \in S} w_y(d(x, y) + d(y, x_y) + d(x_y, x)).$$

This problem can be solved, as noted by Tamir and Halman [42], using a covering argument. Let r be the parameter of the covering problem. To ensure a weighted round-trip cover of r for $y \in S$ one has to consider all the points of the set $Y_y(r) = \{x \in \mathbb{R}^2 : d(x, y) + d(x, x_y) \leq r/w_y - d(y, x_y)\}$. In the Euclidean case, $Y_y(r)$ is an ellipse, and the problem reduces to finding the smallest value of r such that $\bigcap_{y \in S} Y_y(r) \neq \emptyset$. Since the ellipses can be in general position the problem is not separable and therefore it does not reduce to Dyer's model. Nevertheless, this Euclidean problem can be solved in $O(n)$ time by invoking the algorithm by Chazelle and Matoušek [10].

We note that in the rectilinear case the above round-trip model reduces to a 3-variable linear program, and therefore can be solved in $O(n)$ time, (see [42]).

Summarizing we have the following result.

Proposition 6.7 *The rectilinear and Euclidean round-trip planar single facility center problems are solvable deterministically in linear time.*

We are unaware of any algorithms in the literature to solve the discrete version of the above round-trip problem. (Recall from the Introduction and Sect. 2.1, that in the discrete version of a center problem, the location of the center is restricted to be in some prespecified finite set $V = \{v_1, \ldots, v_m\}$.) We next show that the discrete problem can be formulated as a special case of the extended parametric model in Sect. 2 with the constraints, $H_i(x_1, x_2 : z) \leq 0$, $i = 1, \ldots, n$.

Using the above notation, for each $y \in S$, define $v_y = y$, $u_y = x_y$, and $s_y = w_y d(y, x_y) = w_y d(v_y, u_y)$. Also set $s = \max_{y \in S} s_y$, and $z' = z - s$. Then, the constraint $w_y(d(x, v_y) + d(x, u_y) + d(v_y, u_y)) \leq z$, is equivalent to

$$w_y d(x, v_y) + w_y d(x, u_y) \leq (z' + s - s_y).$$

Notice that since $z \geq s$, we have $z' \geq 0$. The set $E'_y := \{x \in \mathbb{R}^2 : d(x, v_y) + d(x, u_y) = \frac{(z'+s-s_y)}{w_y}\}$ is an ellipse whose foci are v_y and u_y. ($\frac{(z'+s-s_y)}{w_y}$ is the length of the axis of E'_y containing these two foci.)

To obtain a function $H_y(x_1, x_2 : z)$ satisfying the three conditions in Sect. 2, we first assume without loss of generality, that the axes of the ellipse E'_y are parallel to the coordinate axes and it is centered at $(0, 0)$. (Any other ellipse with the same axes lengths can be obtained from this ellipse by applying a rotation and a translation.)

Let E be an ellipse with foci $v_y = (-f, 0)$ and $u_y = (f, 0)$, (i.e., its axes are parallel to the coordinate axes), centered at $x = (0, 0)$. An analytical expression of

E, given that its horizontal (major) axis has length $2h(z')$ with $2h(z') = \frac{(z'+s-s_y)}{w_y}$, is

$$E = \{x \in \mathbb{R}^2 : d(x, v_y) + d(x, u_y) = 2h_y(z')\}.$$

The length of its vertical (minor) axis, denoted by $2b$, is then defined by $f^2 = [h_y(z')]^2 - b^2$. By using basic algebraic transformations, (see [7]), we have,

$$E = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : \frac{x_1^2}{[h_y(z')]^2} + \frac{x_2^2}{[h_y(z')]^2 - f^2} = 1 \right\}.$$

Therefore, the inequality $d(x, v_y) + d(x, u_y) \le 2h_y(z')$ is equivalent to $H_y(x_1, x_2 : z') \le 0$, where $H(x_1, x_2 : z')$ is defined by

$$H(x_1, x_2 : z') = x_1^2([h_y(z')]^2 - f^2) + x_2^2[h_y(z')]^2 - [h_y(z')]^2([h_y(z')]^2 - f^2).$$

It is easy to check that $H_y(x_1, x_2 : z')$ satisfies the three conditions listed in Sect. 2.

1. $H_y(x_1, x_2 : z')$ is a polynomial of degree 4 of its three variables, (x_1, x_2, z').
2. For each real z', $H_y(x_1, x_2 : z')$ is a convex quadratic function of (x_1, x_2).
3. For $z' \le z''$, $\{(x_1, x_2) : H_y(x_1, x_2 : z') \le 0\} \subseteq \{(x_1, x_2) : H_y(x_1, x_2 : z'') \le 0\}$, since the lengths of the axes of the ellipse $\{(x_1, x_2) : H(x_1, x_2 : z') = 0\}$ are respectively smaller than the lengths of the axes of the ellipse $\{(x_1, x_2) : H(x_1, x_2 : z'') = 0\}$. (The latter follows from the monotonicity of the function h_y, and the fact that the two ellipses have the same pair of foci.)

Therefore, applying Theorem 4.2, we obtain the following result.

Proposition 6.8 *The discrete version of the customer round trip planar single facility location problem can be solved in* $O(m \log^3 n + \lambda_5(n) \log^3 n)$.

7 Final Comments and Questions

In this paper we introduced a unifying model for 1-center planar location problems with transportation costs defined by piecewise quadratic convex functions of the distances. We use the results by Chazelle and Matoušek [10], to claim that the continuous model can be solved deterministically in linear time. For the pure and separable case we also show an alternative $O(n)$ deterministic algorithm based on the model in [18]. (The latter algorithm can be used to solve the piecewise separable version in $O(n \log n)$ time.)

To solve the discrete version we resort to methods of computing envelopes of curves and the parametric approach of Megiddo [31], and solve the problem in subquadratic time. Using our models we improve upon some existing results. Specifically, the continuous model of Foul [22], which was solved there, approximately only, by general nonlinear iterative algorithms, is solved here exactly by an $O(n)$ deterministic algorithm. In addition, we solve the discrete version of Foul's model in $O(\lambda_5(n) \log^3 n)$ time. (In the discrete version of Foul's model the center must be located at one of the planar points in $\{(A_i, B_i) : i = 1, \ldots, n\}$.) The $O(n \log n)$ complexity of the scalarized and constrained models in [36], is improved to $O(n)$ even for the weighted versions.

Appendix

In this Appendix we include, for the sake of readability, several technical results that are used in the paper.

A.1 Complexity of Intersections and Envelopes of Jordan Arcs and Curves

We cite and list several theorems that give the complexity of intersecting Jordan arcs and curves.

Theorem 8.1 [41, Theorem 6.1] *Given a set of n unbounded x_1-monotone Jordan curves with at most s intersections between any pair of curves, its lower envelope has a $\lambda_s(n)$ combinatorial complexity, and it can be computed in $O(\lambda_s(n)\log n)$ time.*

Theorem 8.2 [41, Theorem 6.5] *Given a set of n x_1-monotone Jordan arcs with at most s intersections between any pair of arcs, its lower envelope has an $O(\lambda_{s+2}(n))$ complexity, and it can be computed in $O(\lambda_{s+1}(n)\log n)$ time.*

Theorem 8.3 ([25], [41, Sect. 6.7]) *Given a set of n unbounded x_1-monotone Jordan curves with at most s intersections between any pair of curves, its lower envelope can be computed in parallel in $O(\log n)$ time using $O(\lambda_s(n))$ processors.*

Theorem 8.4 ([25], [41, Sect. 6.7]) *Given a set of n x_1-monotone Jordan arcs with at most s intersections between any pair of arcs, its lower envelope can be computed in parallel in $O(\log n)$ time using $(\lambda_{s+2}(n))$ processors.*

$\lambda_s(n)$ is the maximum length of a Davenport-Schinzel sequence of order s on n symbols. The reader is referred to Chap. 3 in [41] for the exact definitions and properties of the functions $\lambda_s(n)$. We note that $\lambda_1(n) = O(n)$, $\lambda_2(n) = O(n)$, $\lambda_3(n) = \theta(n\alpha(n))$, and $\lambda_4(n) = \theta(n2^{\alpha(n)})$, where $\alpha(n)$ is the inverse of the Ackermann function. Also, for any constant s, $\lambda_s(n) = o(n\log^* n)$.

A.2 Dyer's Class of Convex Programs

Dyer [18] considered the following 'almost' linear program:

$$\begin{aligned} \min \quad & x_1 \\ \text{s.t.} \quad & a_i' x \le b_i, \quad i \in N = \{1, \dots, n\} \\ & x \in K = \{x \in \mathbb{R}^d : g_j(x) \le 0\}, \quad j \in M = \{1, \dots, m\}, \end{aligned} \tag{12}$$

where the functions $g_j(x)$, $j = 1, \dots, m$, are differentiable over some open neighborhood containing K, convex and "essentially equivalent" to polynomials. (For example, $g_j(x) = \sqrt{((x_1 - a_1)^2 + \cdots + (x_d - a_d)^2)}$ or $g_j(x) = -\sqrt{x}_1$.) The maximum degree of these polynomials is denoted by k. In addition, it is assumed that $\max\{m, d, k\} \ll n$.

In his paper Dyer presented a linear time algorithm to solve the above problem. The algorithm, except for some additional technical steps, which are crucial for handling the non-linear constraints, is similar to those of Megiddo [32], Dyer [17] and Clarkson [11]. The output is a set of at most d linear equalities and some nonlinear constraints binding at the optima, which will describe the optimal solutions of (12). Let $T(m,k,d,n)$ be the number of algebraic operations needed to solve (12). This number satisfies $T(m,k,d,n) \le (km)^{O(d)} 3^{d^2} n$. Also, the space bound is polynomial in m,k,d and linear in n. (The term $(km)^{O(d)}$ is the effort needed to solve the problem non-recursively when $n=0$, which is equivalent to checking a polynomial predicate in the variables $x_1,\dots,x_d$. Constructing and checking this polynomial is doable, using the results by Renegar [38, 39]), and Basu et al. [3] in $(mk)^{O(d)}$ time.) For further details the reader is referred to [18].

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